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Wave-induced distortions of a slightly stratified shear flow: a nonlinear critical-layer effect

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A slightly stratified shear flow is considered when the effects of nonlinearity, viscosity and thermal diffusivity are in balance in the critical layer. Finite amplitude essentially non-diffusive neutral waves exist only if the mean temperature, velocity and vorticity profiles are distorted such that small jumps in these quantities occur across the critical layer.

1. Introduction

It is well known that critical layers, regions near which the wave velocity equals the mean flow velocity, are of utmost importance in the study of homogeneous and stratified shear flows, when the Reynolds number of the flow is large. In this paper, the critical layer for a slightly stratified shear flow is investigated for the case in which the effects of viscosity, thermal diffusivity and nonlinearity are in balance.

It is only in the critical layer (and near external boundaries) that the inviscid linearized disturbance equation is not valid. For homogeneous shear flows this singularity was first analysed by incorporating viscous effects in the neighbourhood of the singular point. Lin's (1945, 1955) careful asymptotic analysis for large Reynolds numbers of the Orr–Sommerfeld equation (the linearized viscous disturbance equation) results in a logarithmic phase shift of $-\pi$ (if it is assumed that the critical layer is asymptotically distinct from the boundary layer at any wall). In a similar manner, the stratified problem was studied by Miles (1961, 1963) and Booker & Bretherton (1967). The jumps in wave properties across the critical layer show that waves are damped in the stratified case upon travelling through the critical layer if the Richardson number is greater than $\frac{1}{4}$.

More recently, Benney & Bergeron (1969) and, independently, Davis (1969) developed a theory for homogeneous shear flows in which nonlinear effects rather than viscous effects remedied the singularity in the linearized inviscid disturbance equation. The logarithmic phase shift across the critical layer vanished. On this basis, Benney & Bergeron (1969) calculated new neutral waves. In order to relate these new solutions to those previously obtained, Haberman (1972) permitted viscous and nonlinear effects to be important in the

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critical layer. In particular, the logarithmic phase shift varied monotonically between its viscous and nonlinear values depending only on a local Reynolds number in the critical layer. A small jump across the critical layer in mean vorticity is induced by the finite amplitude neutrally stable wave. Although it vanishes in the limit in which viscous effects are dominant in the critical layer, the jump in mean vorticity persists when nonlinear effects dominate. Here it will be shown that slight stratification causes the finite amplitude wave to induce not only a distortion of the mean vorticity, but also one in mean velocity and mean temperature.

Kelly & Maslowe (1970) extended to stratified shear flows the nonlinear critical-layer analysis developed for non-stratified flows by Benney & Bergeron (1969). The stronger singularity in the stratified problem results in the dynamics of the critical layer being governed by an analytically difficult coupled set of nonlinear partial differential equations. Consequently, Kelly & Maslowe (1970) considered the more tractable slightly stratified problem, which nonetheless incorporates the first non-trivial influence of the stratification. Only the case in which nonlinear effects are dominant in the critical layer was discussed. As in the homogeneous case, the logarithmic phase shift of the disturbances across the critical layer vanished. More recently, the more difficult problem in which the stratification is no longer small has been considered numerically by Maslowe (1972).

2. Formulation

The method of matched asymptotic expansions is employed to determine the dynamics of the critical layer. Using the Boussinesq approximation, Kelly & Maslowe (1970) derived the equations of motion for neutrally stable, finite amplitude, two-dimensional, periodic disturbances to a slightly stratified shear flow when the nonlinear effects are dominant in the critical layer. Here, two modifications to the work of Kelly & Maslowe (1970) are made. (i) The effects of viscosity, thermal diffusivity and nonlinearity are in balance in the critical layer. (ii) Possible small distortions of the mean profiles are taken into account, as was shown to be necessary in the non-stratified case by Haberman (1972). The notation below is that used for nonlinear critical layers by Benney & Bergeron (1969), Kelly & Maslowe (1970), Maslowe (1972) and Haberman (1972).

For flows with small Richardson numbers $J_c = \epsilon^{\frac{1}{2}}\hat{J}_c$ corresponding to a weak stratification, the leading-order streamlines in the critical layer are in the pattern of 'cat's eyes'.

$$u_c'(\tfrac{1}{2}Y^2 + \cos\xi) = \text{constant},$$

where $\xi = \alpha(x-ct)$ and $y-y_c = \epsilon^{\frac{1}{2}}Y$, meaning that the thickness of the critical layer is $O(\epsilon^{\frac{1}{2}})$. For direct comparison with Kelly & Maslowe (1970), this corresponds to an amplitude normalization of the Frobenius solutions of the linearized inviscid equation such that $B = 1$, while for comparison with Haberman (1972) $B = u_c'$.

The weakly coupled linear partial differential equations that determine the important relationships across the critical layer are

$$Y\psi^*_{YY\xi}+\sin\xi\psi^*_{YYY}+(u'_c\hat{J}_c/T'_c)\,T^*_\xi = \lambda_1\psi^*_{YYYY}, \tag{2.1}$$

$$YT^*_\xi+\sin\xi T^*_Y = \lambda_2 T^*_{YY}. \tag{2.2}$$

λ_1 is an inverse local Reynolds number in the critical layer, $\lambda_1 = 1/\alpha Re\epsilon^{\frac{3}{2}}u'_c$. λ_2 is that same quantity scaled by the Prandtl number, $\lambda_2 = \lambda_1/Pr$. The terms containing λ_1 and λ_2 represent respectively the viscous and heat-diffusion effects, which are assumed to be in balance in the critical layer with the effects of non-linearity. This implies that both λ_1 and λ_2 are of $O(1)$, in contrast to the problem solved by Kelly & Maslowe (1970), in which both λ_1 and λ_2 are assumed small. T^* is the $O(\epsilon^{\frac{1}{2}})$ temperature in the critical layer. ψ^* is the $O(\epsilon^{\frac{3}{2}})$ stream function in the critical layer. Equation (2.1) and (2.2) are to be solved with the asymptotic conditions as $Y\to\pm\infty$ obtained by the method of matched asymptotic expansions:

$$\psi^* \sim \tfrac{1}{6}u''_cY^3+H^*_\pm Y^2+u''_cY\log|Y|\cos\xi+L^*_\pm Y+Y(A^*_\pm\cos\xi+C^*_\pm\sin\xi)$$
$$+u'_c\hat{J}_c\cos\xi\log|Y|+B^*_{\frac{3}{2}\pm}\cos\xi+D^*_{\frac{3}{2}\pm}\sin\xi+\ldots \quad\text{as}\quad Y\to\pm\infty, \tag{2.3}$$

$$T^*\sim T'_cY+N^*_\pm+(T'_c/Y)\cos\xi+\ldots \quad\text{as}\quad Y\to\pm\infty. \tag{2.4}$$

The terms $\frac{1}{6}u''_cY^3$ and T'_cY are obtained directly from the Taylor series around the critical point respectively of the mean stream function and the mean temperature. The terms $u''_cY\log|Y|\cos\xi$, $u'_c\hat{J}_c\log|Y|\cos\xi$ and $(T'_c/Y)\cos\xi$ result from the more singular of the two Frobenius solutions of the linearized inviscid and non-diffusive disturbance equation, defined by Kelly & Maslowe (1970) in the case of a small Richardson number. The less singular of the two Frobenius solutions yields the term $Y(A^*_\pm\cos\xi+C^*_\pm\sin\xi)$, where $A^*_\pm$ and $C^*_\pm$ are as yet unknown coefficients. In order to solve the eigenvalue problem and hence determine neutral modes, it is necessary to calculate the jump relationships across the critical layer: $A^*_+-A_-$ and $C^*_+-C^*_-$. The terms $H^*_\pm Y^2$ and $B^*_{\frac{3}{2}\pm}\cos\xi+D^*_{\frac{3}{2}\pm}\sin\xi$, which represent respectively an $O(\epsilon^{\frac{1}{2}})$ jump in mean vorticity and an $O(\epsilon^{\frac{3}{2}})$ jump in the amplitude of the fundamental wave disturbance, are necessary as Haberman (1972) showed. These terms did not appear in either Benney & Bergeron (1969) or Kelly & Maslowe (1970), indicating that their solutions can not be obtained continuously from the diffusive problem by taking a non-diffusive limit. New to this work are $N^*_\pm$ and $L^*_\pm Y$, which will be shown to be necesssarily induced by a finite amplitude wave disturbance interacting with a slightly stratified shear flow. $N^*_\pm$ represents a possible $O(\epsilon^{\frac{1}{2}})$ jump in mean temperature across the critical layer. $L^*_\pm Y$ represents a possible $O(\epsilon)$ jump in mean velocity across the critical layer (the possibility of an $O(\epsilon^{\frac{1}{2}})$ jump in mean velocity has been eliminated by Haberman (1972)).

3. Distortions of the mean profiles

Relationships between quantities above and below the critical layers are derived. In particular, it is shown that there must be an $O(\epsilon^{\frac{1}{2}})$ jump in mean temperature and an $O(\epsilon)$ jump in mean velocity across the critical layer. These distort

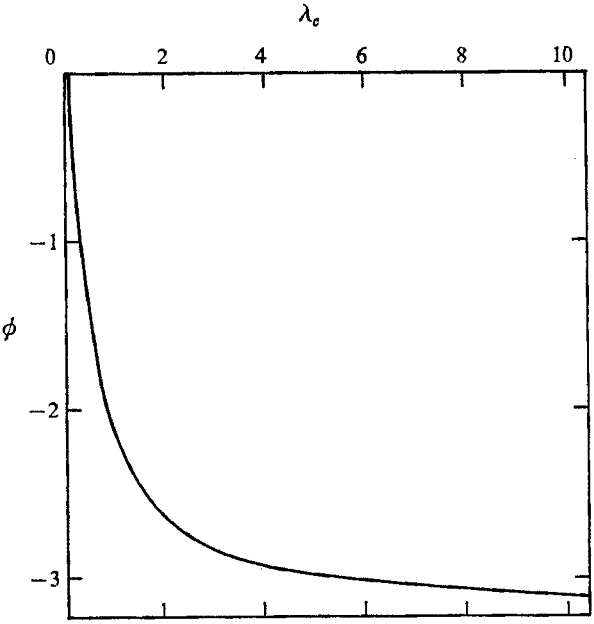

FIGURE 1. The logarithmic phase shift for a homogeneous shear flow.

the original mean profiles from that which presumably existed without the presence of the finite amplitude neutrally stable wave.

Certain results of the investigation by Haberman (1972) of critical layers for a homogeneous shear flow are quite conveniently applicable directly to the stratified problem. In particular, the solution $\psi(\xi, Y; \lambda_c)$ of the following important problem was discussed:

$$Y\psi_{YY\xi} + \sin\xi\psi_{YYY} = \lambda_c\psi_{YYYY}, \tag{3.1}$$

with asymptotic conditions as $Y \to \pm\infty$

$$\psi \sim \tfrac{1}{6}Y^3 + H_\pm Y^2 + Y\log|Y|\cos\xi + A_\pm Y\cos\xi + C_\pm Y\sin\xi$$
$$+ B_{\frac{3}{2}\pm}\cos\xi + D_{\frac{3}{2}\pm}\sin\xi + \dots. \tag{3.2}$$

On the basis of Reynolds stress relationships, some of the jump conditions were derived analytically,

$$D_{\frac{3}{2}+} - D_{\frac{3}{2}-} = 0, \quad B_{\frac{3}{2}+} - B_{\frac{3}{2}-} = -2(H_+ - H_-), \tag{3.3a,b}$$

$$C_+ - C_- = 4\lambda_c(H_+ - H_-), \tag{3.3c}$$

while the others were derived numerically,

$$A_+ - A_- = 0, \quad C_+ - C_- = \phi(\lambda_c). \tag{3.3d,e}$$

$\phi(\lambda_c)$ is the logarithmic phase shift for the homogeneous shear problem, whose dependence on the inverse local Reynolds number λ_c is reproduced in figure 1 from Haberman (1972). Thus,

$$H_+ - H_- = \phi(\lambda_c)/4\lambda_c. \tag{3.4}$$

The $O(\epsilon^{\frac{1}{2}})$ temperature for the stratified shear problem corresponds to the $O(\epsilon^{\frac{1}{2}})$ vorticity for the homogeneous one, since

$$T^* = T'_c \psi_{YY}(\xi, Y; \lambda_2) \tag{3.5}$$

solves (2.2) and the asymptotic conditions (2.4) are satisfied if

$$N^*_+ - N^*_- = 2T'_c(H_+ - H_-) = T'_c \phi(\lambda_2)/2\lambda_2. \tag{3.6}$$

Thus the $O(\epsilon)$ finite amplitude neutrally stable wave has induced an $O(\epsilon^{\frac{1}{2}})$ jump in mean temperature across the critical layer. This jump will persist in the limit in which nonlinear effects are dominant in the critical layer ($\lambda_2 \to 0$), since, as was discussed by Haberman (1972), figure 1 implies that the phase shift tends to zero such that $\phi(\lambda_2)/\lambda_2$ is bounded as $\lambda_2 \to 0$. Furthermore as $\lambda_2 \to 0$ the temperature will be continuous across the edges of the cat's eyes (and constant inside the cat's eyes), but the temperature gradient will be discontinuous, necessitating a thinner boundary layer on the edges of the cat's eyes. This conclusion follows from (3.5), since in the non-stratified case Haberman (1972) showed that the vorticity ψ_{YY} is continuous across the edges (and constant inside), but by implication the derivative of the vorticity ψ_{YYY} is discontinuous. On the other hand as $\lambda_2 \to \infty$, $\phi(\lambda_2)/\lambda_2 \to 0$. Thus this jump in mean temperature can be neglected if the amplitude of the disturbance is sufficiently small so that the linear diffusive critical-layer theory is appropriate.

The local Richardson number in the critical layer, to leading order in an expansion in powers of ϵ, is given by

$$J = (\epsilon^{\frac{1}{2}}\hat{J}_c/T'_c)\, T^*_Y = \epsilon^{\frac{1}{2}}\hat{J}_c \psi_{YYY}(\xi, Y; \lambda_2),$$

where $\epsilon^{\frac{1}{2}}\hat{J}_c$ is the local Richardson number in the critical layer based on the undisturbed velocity and temperature profiles (without the effects of a finite amplitude wave). It does not depend on the stream function ψ^*; only the derivative of the temperature is important. However, (2.4) shows that, far away from the critical layer, the Richardson number essentially remains at its undisturbed value, and is not influenced by the wave-induced distortions. In the critical layer values of ψ_{YYY}, the derivative of the vorticity in the homogeneous case, are needed. When (3.5) is taken into account, examination of the contours of constant vorticity obtained by Haberman (1971) shows the qualitative variation with position and λ_2 of the local Richardson number. For λ_2 not large, the local Richardson number is noticeably smaller in the interior of the cat's eyes. This is expected for small λ_2, since in that limit the temperature is known to be uniform inside the cat's eyes. However, away from the interior of the cat's eyes, the derivative of vorticity in the homogeneous case is such that the largest local Richardson number is increased over its undisturbed value. The largest value occurs near the edge of the cat's eyes. For $\lambda_2 = 0{\cdot}3$, the maximum local Richardson number has increased by approximately 40 %; for $\lambda_2 = 1{\cdot}0$, the maximum local Richardson number has increased by approximately 20 %; while for $\lambda_2 = 10{\cdot}0$, there is no noticeable increase. Better quantitative expressions for this increase require more accurate values of ψ_{YYY} than those obtained by Haberman (1971). However, for $\lambda_2 \to 0$

the derivative of vorticity, to leading order in a λ_2 expansion, can be obtained analytically:

$$\psi_{YYY} = \frac{2^{\frac{1}{2}}\pi|Y|}{\int_0^{2\pi} (\frac{1}{2}Y^2+\cos\xi-\cos X)^{\frac{1}{2}}\,dX},$$

outside the cat's eyes (it is zero inside the cat's eyes). The maximum value given by this formula can be shown to occur along the edge of the cat's eyes in the middle (specifically, at $\xi = \pi$, $|Y| = 2$). This results in the maximum value of the local Richardson number occurring at this point and having the value $\frac{1}{2}\pi\epsilon^{\frac{1}{2}}\hat{J}_c$, or roughly 57 % more than its undisturbed value. Thus one effect of nonlinearity in the critical layer is to increase the maximum value of the local Richardson number.

To obtain the other jump conditions across the critical layer, the equation for the stream function ψ^* must be solved. Determination of the temperature T^* yields

$$Y\psi^*_{YY\xi}+\sin\xi\psi^*_{YYY}+u_c'\hat{J}_c\psi_{YY\xi} = \lambda_1\psi^*_{YYYY}, \tag{3.7}$$

where $\psi = \psi(\xi, Y; \lambda_2)$. Quantities above and below the critical layer are now shown to be related. The technique to do this, employed by Haberman (1972) in the homogeneous case, involves obtaining integrals of (3.7).

Integrating with respect to Y yields

$$\frac{\partial}{\partial\xi}(Y\psi^*_Y-\psi^*)+\frac{\partial}{\partial Y}(\sin\xi\,\psi^*_Y)+\frac{\partial}{\partial\xi}(u_c'\hat{J}_c\psi_Y) = \lambda_1\psi^*_{YYY}+I_1(\xi). \tag{3.8}$$

Since $I_1(\xi)$ is independent of Y, it can be determined by evaluating (3.8) asymptotically as $Y\to\pm\infty$, using (2.3) and (3.2):

$$-2u_c'\hat{J}_c\sin\xi+B^*_{\frac{3}{2}\pm}\sin\xi-D^*_{\frac{3}{2}\pm}\cos\xi+2H^*_\pm\sin\xi+u_c'\hat{J}_c(-A_\pm\sin\xi+C_\pm\cos\xi) = u_c''\lambda_1+I_1(\xi). \tag{3.9}$$

By subtracting the two equations (3.9), it follows that

$$D^*_{\frac{3}{2}+}-D^*_{\frac{3}{2}-} = u_c'\hat{J}_c\phi(\lambda_2),\quad B^*_{\frac{3}{2}+}-B^*_{\frac{3}{2}-} = -2(H^*_+-H^*_-), \tag{3.10a,b}$$

where equations (3.3) have been used. In the limit $\hat{J}_c = 0$, these agree with the results Haberman (1972) derived in the homogeneous case. Since ψ^* is periodic in ξ with period 2π, by integrating (3.8) over that period and over Y, the following is derived:

$$\int_0^{2\pi}\sin\xi\psi^*_Y\,d\xi = \lambda_1\int_0^{2\pi}\psi^*_{YY}\,d\xi-2\pi u_c''\lambda_1Y+I_2, \tag{3.11}$$

where (3.9) has been used to evaluate

$$\int_0^{2\pi} I_1(x)\,dx.$$

I_2 is a constant, which is again determined by the asymptotic evaluation as $Y\to\pm\infty$ of an equation. Hence (3.11) implies that

$$\pi C^*_\pm = 4\pi\lambda_1H^*_\pm+I_2. \tag{3.12}$$

The jump in mean vorticity is thus proportional to the logarithmic phase shift:

$$C_+^* - C_-^* = 4\lambda_1(H_+^* - H_-^*), \tag{3.13}$$

as was shown in the non-stratified case by Benney & Bergeron (1969) and Haberman (1972). One further integration of (3.11) with respect to Y and another asymptotic evaluation as $Y \to \pm\infty$ shows that

$$D_{\frac{3}{2}+}^* - D_{\frac{3}{2}-}^* = 2\lambda_1(L_+^* - L_-^*). \tag{3.14}$$

Consequently (3.10*a*) and (3.14) imply the existence of an $O(\epsilon)$ jump in mean velocity across the critical layer

$$L_+^* - L_-^* = u_c' \hat{J}_c \phi(\lambda_2)/2\lambda_1. \tag{3.15}$$

In order to maintain a nonlinear neutrally stable periodic wave disturbance to slightly stratified shear flow, the velocity profile must be distorted in the manner indicated by (3.15). As with the temperature profile, (3.15) shows that the distortions of the velocity profile vanish in the linear viscous and diffusive theory. The situation in which $\lambda_1 \ll 1$ and $\lambda_1 \ll \lambda_2$ has an unusual property. This is the case in which the nonlinear effects are dominant in the momentum equation in the critical layer and the Prandtl number is small (it is not necessary that $\lambda_2 \ll 1$). Under these circumstances (3.15) implies that the $O(\epsilon)$ jump in mean velocity can be quite large. The possible consequences of this will not be pursued further here.

In summary, it has been shown that jumps exist across the critical layer in mean temperature and velocity. However, the logarithmic phase shift, so important for any eigenvalue calculation of neutral modes, has not as yet been determined. To do this would require the numerical integration of (2.1) and (2.2) with asymptotic conditions given by (2.3) and (2.4), as was done for the analogous problem for homogeneous shear flows by Haberman (1971, 1972). This calculation will not be performed at this time.

4. Prandtl number equal to one

In the special case in which the Prandtl number is approximately equal to one, such that $\lambda_1 = \lambda_2 \equiv \lambda$, the remaining unknown jump conditions can be obtained analytically. Equation (3.7),

$$Y\psi_{YY\xi}^* + \sin\xi\,\psi_{YYY}^* + u_c'\hat{J}_c\psi_{YY\xi} = \lambda\psi_{YYYY}^*, \tag{4.1}$$

has a particular solution

$$u_c'\hat{J}_c\psi_Y,$$

where $\psi = \psi(\xi, Y; \lambda)$. This result follows from differentiation with respect to Y of (3.1). A more general solution to (4.1) is obtained by superimposing multiples of known homogeneous solutions. Thus

$$\psi^* = u_c'\hat{J}_c\psi_Y + K_0\psi + K_1Y^2 + K_2(\xi)\,Y + K_3(\xi), \tag{4.2}$$

where ψ is given by (3.1)–(3.3). Fortunately, the K_i can be chosen such that not only is (4.2) a solution to (4.1), but it also satisfies the asymptotic conditions (2.3).

In order for (4.2) to have the necessary mean shear $\frac{1}{6}u_c''Y^3$, it follows that $K_0 = u_c''$. Now all the jump relationships are determined by comparing the known asymptotic conditions of $\psi(\xi, Y; \lambda)$ given by (3.2) and (3.3) with the partially unknown asymptotic conditions of ψ^* given by (2.3). In this manner, the following can be derived:

$$L_+^* - L_-^* = u_c'\hat{J}_c\phi(\lambda)/2\lambda, \quad D_{\frac{3}{2}+}^* - D_{\frac{3}{2}-}^* = u_c'\hat{J}_c\phi(\lambda), \tag{4.3a,b}$$

$$B_{\frac{3}{2}+}^* - B_{\frac{3}{2}-}^* = -u_c''\phi(\lambda)/2\lambda, \quad A_+^* - A_-^* = 0, \tag{4.3c,d}$$

$$C_+^* - C_-^* = u_c''\phi(\lambda), \quad H_+^* - H_-^* = u_c''\phi(\lambda)/4\lambda. \tag{4.3e,f}$$

These jump conditions along with (3.6) completely relate the solution above the critical layer to that below the critical layer. Equation (4.3a), for the $O(\epsilon)$ distortion of the mean velocity, and (4.3b) agree respectively with (3.10a) and (3.15) in the case $Pr = 1$. Furthermore (4.3c), (4.3e) and (4.3f) are consistent with the incomplete jump conditions (3.10b) and (3.13). It is noted that conditions (4.3d), (4.3e) and (4.3f) are identical to those obtained in the non-stratified case by Haberman (1972). In particular, the logarithmic phase shift is again determined directly from figure 1. The concept of a logarithmic phase shift is quite helpful since besides (4.3d) and (4.3e), (4.3b) is simply derived if the complex logarithmic function appearing in the Frobenius solutions of the linearized inviscid and non-diffusive equation are interpreted in the following manner:

$$\log(y-y_c) \Rightarrow \begin{cases} \log|y-y_c| & \text{above the critical layer,} \\ \log|y-y_c| + i\phi(\lambda) & \text{below the critical layer.} \end{cases}$$

In the case in which $u_c'' = 0$, $H_+^* = H_-^*$ for example, and (4.2) satisfies the matching condition (2.3) if $K_1 = H_+^* - \frac{1}{2}u_c'\hat{J}_c$.

Again in the limit in which nonlinear effects dominate in the critical layer ($\lambda \to 0$), the analysis for ψ in this limit previously performed by Haberman (1972) shows that (4.2) implies that the velocity is continuous, but that the vorticity is discontinuous at the edge of the cat's eyes (although a non-vanishing constant inside). Thus not only is a thin boundary layer at the edge of the cat's eyes necessary to smooth the discontinuous temperature gradient, but also to smooth the discontinuous vorticity. Furthermore for $\lambda \to 0$, in the case in which $u_c'' = 0$, equation (4.2) implies that the velocity is constant inside the cat's eyes.

5. Discussion

After the jump conditions across the critical layer have been determined, the eigenvalue calculation for neutrally stable waves must still be carried out. However, even after that calculation, the difficult question remains of determining the time-dependent mechanism for the generation of these finite amplitude modes. Solving this requires an understanding of the nonlinear initial-value problem.

The analysis in this paper applies to the relatively simple case in which the Richardson number at the critical level is small. Then small distortions of the mean temperature, velocity and vorticity are necessary for the existence of a neutrally stable finite amplitude wave disturbance. This suggests that the problem of greater interest, studied by Maslowe (1972), when the Richardson number is no longer small (but nonlinear effects dominate), might need to be modified by also allowing jumps in the mean profiles. This would probably influence the dynamics of the critical layer.

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